

A short comment on the supersymmetric structure of Chern-Simons theory in the axial gauge

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Abstract

The topological supersymmetry of the pure Chern-Simons model in three dimensions is established in the case where the theory is defined in the axial gauge.

1 Introduction

One of the characteristic features of the Chern-Simons theory [1, 2], and of more general topological field theories [3], is the supersymmetric structure [4, 5, 6, 7] they possess, in certain gauges, and which is at the origin of their ultraviolet finiteness [6, 7]. This supersymmetry is generated by the BRS operator – and sometimes also by the anti-BRS operator – and by one – or two – operators carrying a Lorentz index. For the three-dimensional Chern-Simons theory such a supersymmetry was known to hold in the Landau gauge [4, 5]. The purpose of this note is to show that it also holds in the homogeneous axial gauge.

The axial gauge is particular interesting for the Chern-Simons theory [8, 9, 10, 11], and presumably for other topological theories, since it is in this gauge that the ultraviolet finiteness is the most obvious due to the complete absence of radiative corrections. The choice of the axial gauge is also relevant for the study of topological theories in manifolds with boundaries [12, 10]. Although the role of supersymmetry in such a geometrical setup still needs a clarification, it is important to establish its presence.

In this paper we treat only the gauge fixed classical theory. However, due to the aforementioned absence of radiative corrections in the axial gauge, our results hold in fact for the full quantum theory.

2 BRS invariance and supersymmetry

The Chern-Simons gauge invariant action in three dimensions is given, in the notation of [6], by

$$\Sigma_{\text{CS}}(A) = -\frac{1}{2} \int d^3x \, \epsilon^{\mu\nu\rho} \text{Tr} \left(A_\mu \partial_\nu A_\rho + \frac{2}{3} g A_\mu A_\nu A_\rho \right) . \quad (2.1)$$

In the homogeneous axial gauge characterized by the constant three-vector n^μ , one has to add the gauge fixing and Faddeev-Popov terms [9, 10, 13, 11]

$$\Sigma_{\text{gf}}(A, d, b, c) = \int d^3x \, \text{Tr}(d n^\mu A_\mu + b n^\mu D_\mu c) , \quad (2.2)$$

where d is the Lagrange multiplier field for the axial gauge condition $n^\mu A_\mu = 0$, c and b are respectively the ghost and antighost fields and $D_\mu \cdot = \partial_\mu \cdot + g[A_\mu, \cdot]$ is the covariant derivative. The gauge group G is assumed to be simple, and all the fields are Lie algebra valued and belong to the adjoint representation:

$$\phi = \phi_a \tau^a , \quad \text{for } \phi = A_\mu, d, b, c ,$$

where the τ^a are the generators of G , normalized in such a way that

$$[\tau^a, \tau^b] = f^{abc} \tau^c , \quad \text{Tr}(\tau^a \tau^b) = \delta^{ab} .$$

By construction the gauge fixed action

$$\Sigma(A, d, b, c) = \Sigma_{\text{CS}} + \Sigma_{\text{gf}} \quad (2.3)$$

is invariant under the nilpotent BRS transformations

$$\begin{aligned} sA_\mu &= -D_\mu c, & sb &= d, \\ sc &= g c^2, & sd &= 0, \end{aligned} \quad (2.4)$$

It is well known that the Chern-Simons action in the covariant Landau gauge, besides the BRS symmetry, has other invariances generated by anticommuting operators, which are Lorentz vectors [4, 5, 6]. In the Landau gauge this vector symmetry has the remarkable property of giving rise, together with the BRS operator, to a superalgebra which, closing on the space-time translations, allows for a supersymmetric interpretation of the model [5]. Indeed, as shown by [6], this supersymmetry implies the existence of a supercurrent which has been the starting point for the proof of its perturbative finiteness.

It is remarkable that the same supersymmetric structure is also present in the homogeneous axial gauge considered here. It is indeed straightforward to check that the action (2.3) is invariant under the following infinitesimal transformations:

$$\begin{aligned} v_\mu A_\nu &= \epsilon_{\mu\nu\rho} n^\rho b, & v_\mu b &= 0, \\ v_\mu c &= -A_\mu, & v_\mu d &= \partial_\mu b. \end{aligned} \quad (2.5)$$

The only difference with the corresponding transformation laws in the Landau gauge [5, 6] is that the partial derivative ∂^ρ in the transformation of A_μ has been replaced by the gauge vector n^ρ .

As announced, these transformations, the BRS transformations (2.4) and the translations ∂_μ obey the supersymmetry algebra

$$s^2 = 0, \quad \{v_\mu, v_\nu\} = 0, \quad \{s, v_\mu\} = \partial_\mu + (\text{field equations}). \quad (2.6)$$

It must be noted here that the above vector supertransformations are different from the ones proposed in [9]:

$$\begin{aligned} \bar{v}_\mu A_\nu &= \epsilon_{\mu\nu\rho} n^\rho c, & \bar{v}_\mu b &= -A_\mu, \\ \bar{v}_\mu c &= 0, & \bar{v}_\mu d &= \partial_\mu c, \end{aligned} \quad (2.7)$$

and which can be obtained from (2.5) just by interchanging the ghost fields: $b \leftrightarrow c$. That the transformations (2.7) leave the action invariant follows from the fact that the interchange of b and c is a (discrete) symmetry of the theory. For the same reason[14], to the BRS invariance (2.4) corresponds an "anti-BRS" invariance \bar{s} :

$$\begin{aligned} \bar{s}A_\mu &= -D_\mu b, & \bar{s}b &= g b^2, \\ \bar{s}c &= d, & \bar{s}d &= 0. \end{aligned} \quad (2.8)$$

However, whereas in the Landau gauge the corresponding operators s , \bar{s} , v_μ , \bar{v}_μ and the translation operator ∂_μ form an $N = 2$ supersymmetry algebra [5], in the present case they don't give rise to a closed algebra. For that reason we focus only on the smaller $N = 1$ algebra (2.6) of the operators s , v_μ and ∂_μ .

We remark that both supersymmetries v_μ and \bar{v}_μ originate from the field equations for the components of the gauge field which are tranverse with respect to the gauge vector n^μ . These equations are obtained from the field equation for A_ρ , contracted with the $\varepsilon_{\mu\nu\rho}$ tensor and the gauge vector n^ν . They take the form of a linearly broken Ward identity, after use of the gauge condition

$$\frac{\delta}{\delta d} \Sigma = n^\mu A_\mu , \quad (2.9)$$

where Σ is the gauge fixed action (2.3). They read

$$\left(\epsilon_{\mu\nu\rho} n^\nu \frac{\delta}{\delta A_\rho} + D_\mu \frac{\delta}{\delta d} \right) \Sigma = n^\nu \partial_\nu A_\mu . \quad (2.10)$$

The invariance of the action under the two supersymmetries (2.5) and (2.7) is easily recovered by looking on the functional identities obtained by multiplying (2.10) either by the antighost field b or by the ghost field c , and then by integrating on space-time.

3 Functional identities and off-shell algebra

BRS invariance as well as supersymmetry may be expressed by functional identities upon the introduction of the external fields γ^μ and τ coupled to the nonlinear BRS transformations of A_μ and c , respectively, *i.e.*, upon the addition, to the action (2.3), of the terms

$$\Sigma_{\text{ext}} = \text{Tr} \int d^3x \left(-\gamma^\mu D_\mu c + g \tau c^2 \right) . \quad (3.1)$$

Thus the complete classical action

$$\Sigma(A, d, b, c, \gamma, \tau) = \Sigma_{\text{CS}} + \Sigma_{\text{gf}} + \Sigma_{\text{ext}} , \quad (3.2)$$

obeys the Slavnov identity

$$\mathcal{S}(\Sigma) = \text{Tr} \int d^3x \left(\frac{\delta \Sigma}{\delta A_\mu} \frac{\delta \Sigma}{\delta \gamma^\mu} + \frac{\delta \Sigma}{\delta c} \frac{\delta \Sigma}{\delta \tau} + d \frac{\delta \Sigma}{\delta b} \right) = 0 . \quad (3.3)$$

for BRS invariance, and the broken Ward identity

$$\mathcal{V}_\mu \Sigma = \Delta_\mu^{\text{cl}} \equiv \int d^3x \text{Tr} \left(-\gamma^\nu \partial_\mu A_\nu + \epsilon_{\mu\nu\rho} \gamma^\nu n^\rho d + \tau \partial_\mu c \right) , \quad (3.4)$$

where

$$\mathcal{V}_\mu \equiv \text{Tr} \int d^3x \left(\epsilon_{\nu\mu\rho} (\gamma^\rho - n^\rho b) \frac{\delta}{\delta A_\nu} - A_\mu \frac{\delta}{\delta c} + \partial_\mu b \frac{\delta}{\delta d} - \tau \frac{\delta}{\delta \gamma^\mu} \right) , \quad (3.5)$$

for the supersymmetry (2.5). The term Δ_μ^{cl} in the right-hand-side of (3.4), being linear in the quantum fields, represents a classical breaking which will be left unchanged by the renormalization. This feature is commun to a large class of topological theories [6, 15, 7].

Again supersymmetry, expressed by the broken Ward identity (3.4), follows from the local Ward identity (2.10) which, in the presence of the external fields, takes the form

$$\mathcal{D}_\mu(x)\Sigma = n^\nu \partial_\nu A_\mu + g\varepsilon_{\mu\nu\rho} n^\nu \{\gamma^\rho, c\} , \quad (3.6)$$

with

$$\mathcal{D}_\mu(x) \equiv \epsilon_{\mu\nu\rho} n^\nu \frac{\delta}{\delta A_\rho} + D_\mu \frac{\delta}{\delta d} . \quad (3.7)$$

In order to see this, one has to multiply (3.6) by b , integrate over space-time, and use the gauge condition

$$\frac{\delta\Sigma}{\delta d} = n^\mu A_\mu , \quad (3.8)$$

as well as the equation

$$\frac{\delta\Sigma}{\delta c} + g \left[b, \frac{\delta\Sigma}{\delta d} \right] = n^\nu \partial_\nu b - \partial_\mu \gamma^\mu - g[A_\mu, \gamma^\mu] + g[c, \tau] . \quad (3.9)$$

The latter identity is a local form, valid in the axial gauge, of the ghost equation of [16].

The on-shell algebra given by the anticommutators (2.6), together with BRS nilpotency, is promoted to an off-shell nonlinear algebra. Given an arbitrary functional F of the fields A_μ , d , b , c , γ^μ and τ , this algebra reads

$$\begin{aligned} \mathcal{S}_F \mathcal{S}(F) &= 0 , & \{\mathcal{V}_\mu, \mathcal{V}_\nu\} &= 0 , \\ \mathcal{V}_\mu \mathcal{S}(F) + \mathcal{S}_F(\mathcal{V}_\mu F - \Delta_\mu^{\text{cl}}) &= \mathcal{P}_\mu F , \end{aligned} \quad (3.10)$$

where \mathcal{P}_μ is the translation Ward operator

$$\mathcal{P}_\mu = \int d^3x \sum_{\text{All fields}} \partial_\mu \phi \frac{\delta}{\delta \phi} ,$$

and \mathcal{S}_F is the F -dependent linearized Slavnov operator

$$\mathcal{S}_F = \text{Tr} \int d^3x \left(\frac{\delta F}{\delta A_\mu} \frac{\delta}{\delta \gamma^\mu} + \frac{\delta F}{\delta \gamma^\mu} \frac{\delta}{\delta A_\mu} + \frac{\delta F}{\delta c} \frac{\delta}{\delta \tau} + \frac{\delta F}{\delta \tau} \frac{\delta}{\delta c} + d \frac{\delta}{\delta b} \right) . \quad (3.11)$$

Moreover, if the functional F is a solution of the Slavnov identity (3.3) and of the supersymmetry Ward identity (3.4), then the linearized Slavnov operator and the supersymmetry Ward operator obey the anticommutation rules

$$\mathcal{S}_F \mathcal{S}_F = 0 , \quad \{\mathcal{S}_F, \mathcal{V}_\mu\} = \mathcal{P}_\mu . \quad (3.12)$$

4 Conclusions

We have shown that the Chern-Simons theory in the axial gauge is not only invariant under the BRS transformations s and a type of anti-BRS transformations \bar{s} : it also possesses two supersymmetries v_μ and \bar{v}_μ . Moreover the two sets of operators $\{s, v_\mu\}$ and $\{\bar{s}, \bar{v}_\mu\}$

generate two separate superalgebras of the Wess-Zumino, $N = 1$, type (see (2.6)). They are related by a discrete transformation which consists in the interchange of the ghost and of the antighost fields. However, contrarily to what happens in the covariant Landau gauge, their union does not generate a $N = 2$ supersymmetry algebra.

We have concentrated on the algebra generated by $\{s, v_\mu\}$, writing the corresponding Ward identities and the off-shell algebra obeyed by the functional generators. We have also found that the supersymmetry v_μ results from a local Ward identity (see (3.6)) which is peculiar to the axial gauge.

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